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The gauge Potts model on a generalized Bethe lattice

A Z Akheyan and N S Ananikian

Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan, 375036, Armenia

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Abstract. The Q-states gauge Potts model is exactly solved on a special infinite-dimensional lattice which is a Bethe lattice of plaquettes. The critical properties are studied. The model exhibits a first-order transition for any number of states Q and coordination number $\gamma+1$. The critical values of coupling constant β_c is shown to be in a good agreement with known results.

1. Introduction

An exact solution of lattice gauge theories, as is known, encounters great difficulties. This is mainly due to the non-trivial topological structure of real 3D and 4D lattices (see e.g. Ananikian and Ismailian 1985). In such a situation the development of various approximation methods becomes very important, and in recent years remarkable results have been achieved in this way (see Drouffe and Zuber 1983, Creutz 1988). However, more of known approximations are either not accurate enough or have limited analytical abilities.

For all that, we wish to suggest a new method (Ananikian *et al* 1989, Ananikian and Akheyan 1990). An exact analytical solution is obtained on a special simplified lattice which is a generalization of the Cayley tree. The lattice has trivial structure (it has no closed surface) and generally speaking is nothing but a topological abstraction. However, under certain conditions the results obtained on it can be regarded as quite a good approximation for normal lattices. At the same time the use of analytical expressions permit a deep understanding of the nature of phase transitions.

As an example, we consider the Q-state gauge Potts model (Kogut 1980), since being one of the simplest models it has interesting critical behaviour. Besides the known duality properties of the Potts models we can compare our results with quite trustworthy ones (exact for $d=4$ and obtained from the spin models for $d=3$), in order better to verify the approximation used.

In section 2 we introduce the lattice and briefly discuss its main features. The model is formulated and some analytical expressions are derived in section 3. In section 4 we observe critical properties and present some numerical results. Some further remarks are made in the conclusion.

2. The lattice

The lattice we introduce is constructed by a successive building up of shells. As a zero shell we take the central plaquette, and all the subsequent shells come out by gluing up γ new plaquettes to each free link of a previous shell. As a result we get the Cayley tree of plaquettes (see figure 1) which is characterized by a coordination number (the

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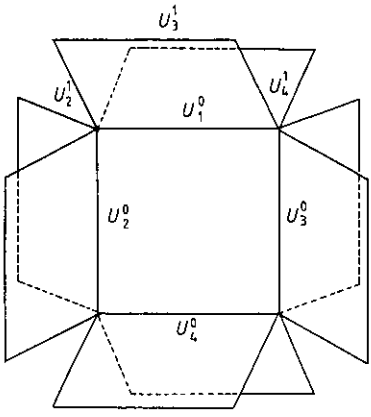

Figure 1. The Cayley lattice of plaquettes with $\gamma = 2$, containing two shells.

number of plaquettes coming out from one link) $\gamma + 1$ and has an infinite Hausdorff dimension.

Initially there is one essential difference between this lattice and standard ones. Because of the too large number of boundary plaquettes the surface effects on the generalized Cayley tree cannot be neglected even in the thermodynamic limit $n \to \infty$ (n is the number of shells). This leads to an obvious inequality of plaquettes belonging to different shells. As a consequence the local parameters, defined on the Cayley tree of plaquettes, will not be translationally invariant.

The last should be considered as a main property of real lattices, and if we want to reproduce their features, we have to restore this invariance. The same problem arises on the usual Cayley tree of links and ways of overcoming it are known (Baxter 1982). From the whole lattice one should turn to studying only the local properties of plaquettes lying deep inside the lattice (infinitely far from the bounary when $n \to \infty$). These plaquettes are entirely equivalent, and the set of them we will call the generalized Bethe lattice by analogy with spin models.

Considering the usual Bethe lattice is known to be equivalent to applying the Bethe–Peierls approximation (Bethe 1935, Peierls 1936). Thus the given approach can be considered as a development of the Bethe approximation for gauge theories.

This is not the first attempt of such a kind. Itzykson *et al* (1983) and Zuber (1984) successfully used the Cayley tree of cubes for a partial resummation of the strong coupling diagrams. The authors refuse the lattice of plaquettes because of a possible contradiction with the Elitzur theorem (Elitzur 1975). Indeed in our study we have to introduce at some stage a gauge non-invariant quantity x, which is forbidden by the Elitzur theorem. However, Maslanka (1988) showed that the latter theorem is not valid for infinite dimensional models, and such are in fact the models on our lattice.

Finally we wish to mention that considering only internal plaquettes does not mean that we can neglect the boundary conditions. The latter play an important role in pure gauge theories, making possible the ordered state. How this works in detail we shall see in section 4.

3. The model

In the Q-state gauge Potts model field variables U_{ij} defined on the links $\langle ij \rangle$ take their values among the group of the Q roots of unity $U_{ij} \in Z$. Then the action is written in

the form

$$S=\beta\sum_{\mathrm{pl}}\delta_{U_{\mathrm{pl}},1} \tag{3.1}$$

where $U_{\mathrm{pl}}=U_{ij}U_{jk}U_{kl}U_{li}$ is the product of the gauge variable along the plaquette contour; δ is the Kronecker symbol; β is the gauge coupling constant, and the sum is taken over all the plaquettes of the lattice.

Summing over all possible configurations of the field variables $\{U\}$ we find a partition function of the model:

$$\mathscr{Z}=\sum_{\{U\}}\exp S \tag{3.2}$$

and the free energy per plaquette is

$$f=\frac{1}{N_{\mathrm{pl}}}\ln\mathscr{Z} \tag{3.3}$$

here N_{pl} is the total number of lattice plaquettes.

We will also use the so-called 'average plaquette':

$$P=\langle\delta_{U_{\mathrm{pl}},1}\rangle=\mathscr{Z}^{-1}\sum_{\{U\}}\delta_{U_{\mathrm{pl}},1}\exp S \tag{3.4}$$

which is the simplest gauge-invariant quantity. It is connected with the free energy via the relation

$$P=-\frac{\partial f}{\partial\beta} \tag{3.5}$$

and thus describes one plaquette's internal energy.

Before proceeding with our analyses, we would like to note that all calculations given below are done initially for the central link and central plaquette. Based on the translational invariance we consider them true for all neighbouring links and plaquettes (lying at the finite distance) and this is the very moment when we neglect the surface effects and turn from the whole lattice to its internal part.

The model is solved by means of exact recursion equations. On the finite lattice with n shells the partition function (3.2) may be rewritten in the form

$$\mathscr{Z}=\sum_{U_i^0}\exp(\beta\delta_{U_{\mathrm{pl}}^0,1})[g_n(U_1^0)]^\gamma[g_n(U_2^0)]^\gamma[g_n(U_3^0)]^\gamma[g_n(U_4^0)]^\gamma \tag{3.6}$$

where the first exponent is the contribution of the central plaquette and $g_n(U_i^0)$ denote the partition function of one branch starting from the zero shell link U^0.

Similarly $g_n(U_i^0)$ can be expressed through $g_{n-1}(U_i^1)$—partition function of the branch, containing $n-1$ shells and starting from the first shell link U^1:

$$g_n(U^0)=\sum_{U_1^1,U_2^1,U_3^1}\exp(\beta\delta_{U_{\mathrm{pl}}^1,1})[g_{n-1}(U_1^1)]^\gamma[g_{n-1}(U_2^1)]^\gamma[g_{n-1}(U_3^1)]^\gamma. \tag{3.7}$$

Introducing the notation

$$x_n\equiv\frac{g_n(U\neq 1)}{g_n(U=1)} \tag{3.8}$$

and having summed expression (3.7) over U^1 we obtain for x_n a recursion formula

$$x_n=Y(x_{n-1})$$

where

$$Y(x,\beta)=\frac{(e^{\beta}+a)(ax^{\gamma}+1)^3-(1-x^{\gamma})^3(e^{\beta}-1)}{(e^{\beta}+a)(ax^{\gamma}+1)^3+a(e^{\beta}-1)(1-x^{\gamma})^3}. \tag{3.9}$$

Here and below $a\equiv Q-1$ notation is used for writing the formulae in a more compact form.

In the thermodynamic limit $n\to\infty$ the recursion succession $\{x_n\}$ tends (as known) to the stable solution of the equation

$$x=Y(x,\beta) \tag{3.10}$$

if the latter exist. Equation (3.10) is in fact the equation of state, since though x has no direct physical meaning, one can express through it all thermodynamic parameters. Thus, for example, the average plaquette calculated from (3.4) is equal to

$$P=e^{\beta}\frac{(ax^{\gamma}+1)^4+a(1-x^{\gamma})^4}{(ax^{\gamma}+1)^4(e^{\beta}+a)-a(1-x^{\gamma})^4}. \tag{3.11}$$

Free energy can be determined from (3.3) by integration:

$$f=-\int P(\beta)\,d\beta \qquad C=-\beta+\int[1-P(x,\beta(x))]\left(\frac{d\beta}{dx}\right)dx+C. \tag{3.12}$$

Substituting $P(x,\beta)$ from (3.11) and $\beta(x)$, $d\beta/dx$ from the equation of state (3.10) (see equation 4.1) we find

$$\begin{aligned}f=&-\beta-3\ln|ax^{\gamma}+1|-3\ln|x^{\gamma}-1|+\frac{\gamma-1}{\gamma+1}\ln|ax^{\gamma+1}+1|\\&-\ln|a(a+1)x^{3\gamma+1}-(a^2-a+1)x^{3\gamma}+3ax^{2\gamma+1}\\&-3(a-1)x^{2\gamma}-3x^{\gamma}+x|+C.\end{aligned} \tag{3.13}$$

This defines the free energy per plaquette up to the integration constant C. The latter cannot be uniquely determined; in general it has different values in each phase. This will be discussed in more detail in section 4.

4. Critical properties

Analysis of critical behaviour is based on the study of solutions of the state equation (3.10). For this it is more convenient to use the plot of function $\beta(x)$ (figure 2(*a*)), since each point of this plot corresponds to some solution of equation (3.10). At $Q=2$ (Z_2 gauge model) function $\beta(x)$ has two branches in regions $x<1$ and $x>1$, but they are equivalent due to the gauge transformation ($x\to 1/x$ in this case). At $Q>2$ there are no solutions $x>1$ (except the cases mentioned at the end of this section), so it is enough to consider only the region $0\leqslant x\leqslant 1$.

As we see, $x=1$ is one of the solutions of the state equation at any value of β. Analysis of the recursion formula (3.9) shows that it is a stable limiting point for succession $\{x_n\}$, hence $x=1$ describes one of possible phases of the system. So far as $x=1$ is the only solution of (3.10) at small $\beta<\beta_{sp}$, this will be the strong coupling phase. One can also call it the disordered phase, since we have $g_n(U=1)=g_n(U\neq 1)$.

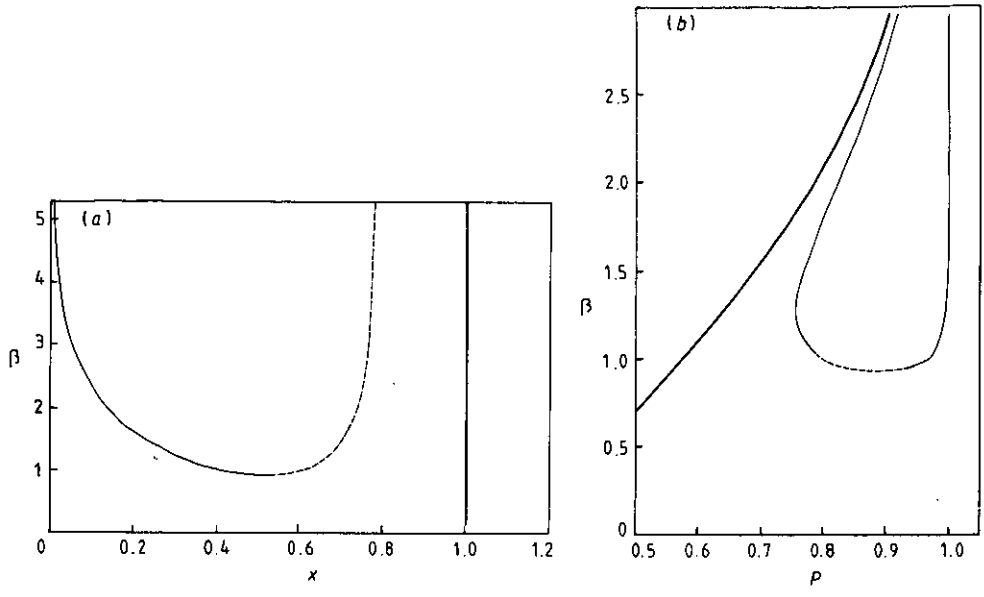

Figure 2. (*a*) The plot of β as a function of x for $Q=3$ and $\gamma=5$. (*b*) The plot of β as a function of P for the same values of Q and γ. The thick full lines represent the strong coupling phase; the thin full lines represent the weak coupling phase; the broken lines correspond to the unstable solutions of the equation of state.

Starting with $\beta=\beta_{sp}$ there appear two new solutions x' and x_{II}. The first, x', is not a stable point for the recursion succession and therefore no physical state corresponds to it. As for solutions $x_{II}(\beta)$, they are stable and describe the second phase of the model—the weak coupling phase.

The appearance of a new phase does not yet mean the existence of phase transition. First one should solve the outstanding problem of the values of the gauge variable U ($x_{II}\neq 1$, hence the new phase will be disordered $g_n(U=1)\neq g_n(U\neq 1)$). This problem is closely connected with the choice of boundary conditions. At a free boundary all terms of the recursion succession x_n will be equal to 1, and only the disordered state $x=1$ is possible. Fixing all boundary variables on some value, say $U^b=1$, will shift the succession $\{x_n\}$ from the point $x=1$ and make feasible a transition to the disordered state.

The behaviour of function $\beta(x)$ is much the same for any γ and Q. In all cases the model undergoes only a first-order transition, since the new solution x_{II} does not coincide with the older one $x_I=1$ even in the appearing point β_{sp}. The latter is not the point where transition occurs, but only the spinoidal point which is the limiting metastable region. One can find the value of β_{sp}, x_{sp} from the condition

$$\left(\frac{d\beta}{dx}\right)_{x=x_{sp}}=0 \tag{4.1}$$

which is reduced to the equation on x_{sp}:

$$(x^\gamma-1)(ax^\gamma+1)-3\gamma x^{\gamma-1}(x-1)(ax+1)=0. \tag{4.2}$$

The actual location of the first-order phase transition should be deduced from comparison of the free energies of both phases:

$$f(x_I,\beta_c)=f(x_{II},\beta_c). \tag{4.3}$$

In this comparison the as yet undetermined constant in (3.13) becomes of some relevance. We must find it independently in each phase, taking as a basis the original definition (3.3) and considering the corresponding small and large coupling limits.

(i) *Strong coupling.* Substituting $x=1$ in equation (3.11) we obtain for the average plaquette:

$$P_s=\frac{e^{\beta}}{e^{\beta}+1}. \tag{4.4}$$

Then free energy, calculated from (3.12), takes the form

$$f=\ln(e^{\beta}+a)+C_s. \tag{4.5}$$

In $\beta\to 0$ limit we have

$$f(\beta\to 0)=\ln(a+1)+C_s. \tag{4.6}$$

On the other hand from (3.3) we have

$$f=\frac{1}{N_{pl}}\ln\left[\sum_{\{U\}}\exp\left(\sum_{pl}\beta\delta_{U_{pl},1}\right)\right]\to\frac{N_l}{N_{pl}}\ln(a+1) \tag{4.7}$$

where N_l/N_{pl} is the number of links/plaquettes of the lattice. Comparing (4.6) and (4.7) we see that

$$C_s=\left(\frac{N_l}{N_{pl}}-1\right)\ln Q. \tag{4.8}$$

(ii) *Weak coupling.* From equation (3.13) in the limit $\beta\to\infty$ we obtain

$$f(\beta\to\infty)=\beta+C_w. \tag{4.9}$$

At the same time $\beta\to\infty$ limit selects only the gauge fields configurations maximizing S_g, i.e. with all U_{pl} set to unity. The number of such configurations is Q^{N_s} (N_s is the number of lattice sites), hence

$$f\to\beta+\frac{N_s}{N_p}\ln Q. \tag{4.10}$$

Thus the weak coupling constant will equal

$$C_w=\frac{N_s}{N_p}\ln Q. \tag{4.11}$$

Now we can compare our results with exact or approximate ones obtained on real lattices. As far as effective dimensionality of the generalized Bethe lattice appears to be $d=\infty$, we expect to have more accuracy in high dimensions. An this is in fact the case. $d=3$ simple cubic lattice can be simulated in this approach by setting

$$\gamma=3 \qquad C_s=0 \qquad C_w=\tfrac{1}{3}\ln Q. \tag{4.12}$$

Corresponding results for critical coupling β_c are presented in table 1, in comparison with more reliable Monte Carlo data. We observe exactness up to 5%. Meanwhile accuracy improves with increasing Q.

The situation is quite different in four dimensions. On $d=4$ hypercubic lattice gauge Potts models are self-dual, and the critical point can be deduced exactly: $\beta_c=\ln(1+\sqrt{Q})$. We represent this model when

$$\gamma=5 \qquad C_s=-\tfrac{1}{3}\ln Q \qquad C_w=\tfrac{1}{6}\ln Q. \tag{4.13}$$

Table 1. Critical values of the coupling constant β at $d=3$.

Q	Ours	MC†
2	1.459	1.523‡
3	1.554	1.626§
4	1.626	1.716¶
6	1.799	1.850‖

† Monte Carlo results (from dual spin Potts models).
‡ Blote *et al* (1989).
§ Fukugita and Okawa (1989).
¶ Chin-Kun and Kit-Sing (1989).
‖ Wu (1982).

Now the error is only about 0-1% (see table 2), which is much better than correspoding mean field results. Again we observe the obvious improvement with increasing Q.

Finally we wish to mention one more feature of the model considered. As we have said, at $Q=2$ all expressions are invariant under $x \to 1/x$ transformation. For $Q>3$ there is no such invariance and equation (3.10) does not have solutions at $x>1$. But this is not the whole truth. Actually at sufficiently large γ function $\beta(x)$ has a second branch in the region $x>1$ (figure 3). These solutions are not equivalent to those of

Table 2. Critical values of the coupling constant β at $d=4$.

Q	MF†	Ours	Exact
2	0.672	0.863	0.881
3	0.800	0.995	1.005
4	0.911	1.091	1.099
5	0.966	1.169	1.174
6		1.234	1.238
8		1.340	1.343
10	1.276	1.424	1.426

† Mean field results (Camarata *et al* 1984).

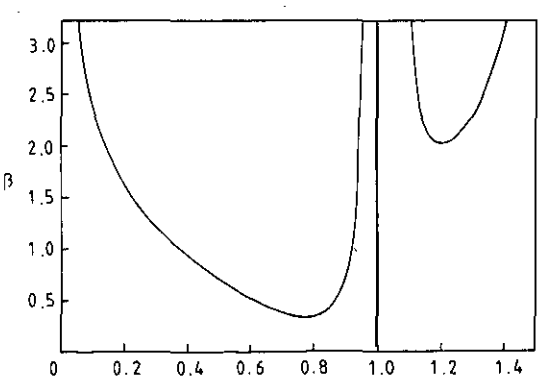

Figure 3. β as a function of x for $Q=3$, $\gamma=14$. The appearance of a new solution is seen in the region $x>1$.

the region $0 < x < 1$ and stable ones describe new states of the system. We can even find the second first transition point. However, the analysis of free energy shows that this transition takes place in a metastable region.

5. Conclusion

As we have mentioned, using the generalized Bethe lattice can be considered as an extension of Bethe–Peierls approximation for gauge models. In spin theories the Bethe approximation is known to belong to the same class of approximation as mean field theory (MFT), being an improvement of the latter. In practice this means that both these approaches must coincide in their prediction of order of phase transition and values of critical exponents. As for the other critical parameters, the Bethe approximation gives better results than MFT (see e.g. Perrugi *et al* 1983).

It seems to be true for gauge theories as well. MFT is known always to predict for gauge models a first-order transition. The same prediction is valid in our approach, though the critical values of the coupling constant β coincide much better with known results, especially for $d \geq 4$.

This coincidence improves with increasing Q. However, the limit $Q \to \infty$ does not give us anything reasonable. Camarata *et al* (1984) mentioned a net difference between Z_Q and Potts gauge models. The former with a limit of large Q can be associated with U(1) gauge models, while the latter do not go into any model possessing a continuous symmetry group when $Q \to \infty$. This statement is completely confirmed in our model.

We restrict our consideration to pure gauge Potts models. As a further extension of this model one can introduce matter (Higgs) fields σ_i in the sites of the lattice:

$$S = \beta_g \sum_{pl} \delta_{U_{pl},1} + \beta_m \sum_{\langle ij \rangle} \delta_{\sigma_i U_{ij} \sigma_j, 1}.$$

In this way we found (Ananikian and Akheyan 1991) the line of first-order transitions. This line separates confinement and Higgs phases and terminates by the critical point of second-order transition, so that these two phases are continuously connected. At the same time we failed to obtain the third phase of the system—the free charges phase. It seems that here we reached the limit of our approach: it is not appropriate for large β_m.

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